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DISCONTINUOUS BIFURCATION STATES FOR ASSOCIATED SMOOTH PLASTICITY AND DAMAGE WITH ISOTROPIC ELASTICITY

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Abstract—For many constitutive equations the tangent tensor consists of a rank one modification to the isotropic elasticity tensor with a total of two elasticity parameters and one parameter describing the current state of inelasticity. For small deformations, general expressions are derived for the loss of ellipticity, the corresponding normal to the bifurcation plane and the mode of discontinuous bifurcation for the velocity gradient. If the principal basis of an evolution tensor is used, the current stress or strain state is characterized by two additional parameters. The small number of material and state parameters makes it feasible to use contour plots to illustrate all possible combinations that can provide a discontinuous bifurcation. These bifurcation maps can be used to illustrate the bifurcation properties of a particular plasticity or continuum damage constitutive model. Conversely, the bifurcation maps can be used in conjunction with experimental data on bifurcation features to assist in the development of constitutive equations that provide the correct failure criterion for a given material under all possible stress paths. Published by Elsevier Science Ltd.

1. INTRODUCTION

There are several methods currently being used for defining material failure based on the structure of the equations used to define material behavior. One approach is to determine under what conditions a jump in the velocity gradient exists subject to the condition of compatibility and equilibrium. The resulting discontinuous bifurcation can be interpreted as the initiation of a band of large, localized deformation of which a special case is a shear band. A second approach, based on a perturbation analysis, shows that stability is lost when the lowest eigenvalue of the acoustic tensor ceases to be positive. Yet another approach is to determine when the governing differential equation changes type such as the loss of ellipticity for the equations of equilibrium. All three conditions; the existence of a discontinuous bifurcation, the loss of positiveness of the lowest acoustic eigenvalue, and the loss of ellipticity are equivalent. Because of this coalescence of criteria, it seems appropriate to define the initiation of material failure as that state at which any one of these criteria is first met. Then, analytical procedures which can conveniently identify the existence and properties of the initiation of material failure will be particularly useful in conjunction with the evaluation of analytical and numerical solutions, and in the development of constitutive equations. Here, explicit formulas are given and contour plots are provided which show the critical conditions under which material failure can exist for fairly general constitutive equations.

Based on sufficient conditions for uniqueness (Hill, 1958) or a postulate on material stability (Drucker, 1964) a second order work criterion is often used to identify material failure. However, provided boundary constraints such as prescribed displacements are used, a body can be loaded well beyond the point at which the second order work criterion is met. Even with the appearance of softening and inhomogeneous deformations, the solutions can be stable and unique. Furthermore, an analysis based on the existence of a discontinuous bifurcation can provide explanations as to why the same material can apparently display ductile behavior in one structural and loading configuration and brittle behavior in another (Neilsen and Schreyer, 1993).

The identification of a discontinuous bifurcation has received considerable attention (Rudnicki and Rice, 1975; Vardoulakis *et al.*, 1978; Larsson *et al.*, 1991; Runesson *et al.*, 1991; Zbib, 1991; Ottosen and Runesson, 1991a; Peric *et al.*, 1992; Perrin and Leblond, 1993; Miehe and Schröder, 1994; Steinman and Willam, 1994) as has the study of the loss of material stability (Hill, 1962; Ottosen and Runesson, 1991b; Brannon and Drugan, 1993). The approach based on change of equation type is relatively rare (An and Schaeffer, 1992; Schreyer *et al.*, 1994; An and Peirce, 1995; Chen and Sulsky, 1995).

Simo *et al.* (1993) have taken the analysis one step forward and identified a method for introducing a strong discontinuity into a numerical algorithm. The point at which this step is done is exactly the point at which a discontinuous bifurcation is identified.

To provide the correct amount of inelastic volume strain with a plasticity theory, it is often argued that a nonassociated flow rule is required. The result is that the tangent tensor is not symmetric and there is the possibility of flutter instability, another major topic area being investigated extensively (Rice, 1976; Loret, 1992; Bigoni and Willis, 1994; Bigoni and Zaccaria, 1994). With the advent of continuum damage mechanics, it is possible that a combined damage and plasticity formulation will provide good stress–strain correlations without the use of a nonsymmetric formulation. For this reason, and to keep the presentation as simple as possible only symmetric constitutive equations will be considered here so that the possibility of flutter instabilities is not an issue.

The next section summarizes the theoretical development of the analytical formulas for identifying when failure occurs together with the mode of failure. The formulas involve surprisingly few parameters with the consequence that it is feasible to provide the instability maps in Section 3 that show critical values of the material properties. Illustrations of how symmetry restrictions can be imposed on plasticity and damage models to obtain a tangent tensor consisting of the basic rank one modification to the elasticity tensor are given in Section 4.

The importance of the bifurcation maps is that general features can be seen at a glance. For a particular model and a stress regime, it can be immediately determined if the possibility exists for a discontinuous bifurcation, and if so, the mode of failure initiation. The maps can also be used as part of an overall evaluation of a constitutive model when experimental data on failure are available.

2. MATERIAL INSTABILITY

2.1. *Formulation*

For convenience, the theoretical development of Schreyer and Neilsen (1995) is summarized here. Consider a body in equilibrium with the force field and subjected to a velocity perturbation which is zero over the part of the body where displacements are prescribed. The dynamic equation of motion for the perturbation is

$$\dot{\mathbf{s}}\cdot\nabla = \rho\ddot{\mathbf{v}} \tag{1}$$

in which $\mathbf{s}$ denotes the Cauchy stress, ρ the mass density, $(\ )\nabla$ the gradient with respect to the position $\mathbf{x}$, $\mathbf{v}$ the velocity, and a superposed dot implies differentiation with respect to time, t. For small deformations, the tangent modulus tensor, $\mathbf{T}$, maps the total strain rate to the stress rate:

$$\dot{\mathbf{s}} = \mathbf{T}:\dot{\mathbf{e}} = \mathbf{T}:(\mathbf{v}\nabla). \tag{2}$$

The fourth-order tensor, $\mathbf{T}$, satisfies the minor symmetry conditions and, unless explicitly stipulated, major symmetry is also assumed which means that if plasticity is involved, for example, an associated evolution equation for plastic strain is used. Assume the tangent modulus tensor does not vary significantly with $\mathbf{x}$, so that it can be removed from the gradient operator. Then the equation of motion for the disturbance becomes

$$\mathbf{T}:\cdot(\mathbf{v}\nabla)\nabla = \rho\ddot{\mathbf{v}}. \tag{3}$$

Assume a solution in the form of a planar wave with normal $\mathbf{n}$:

$$\mathbf{v} = \mathbf{v}_0\, e^{ik(\mathbf{n}\cdot\mathbf{x}-ct)} \tag{4}$$

in which $\mathbf{v}_0$ is the amplitude, c the wave speed, and k the wave number. Define the acoustic tensor to be

$$\mathbf{A}(\mathbf{n}) = \mathbf{n}\cdot\mathbf{T}\cdot\mathbf{n} \tag{5}$$

in which the dependence on $\mathbf{n}$ is explicitly emphasized. Then a solution to eqn (3) exists if

$$\mathbf{A}\cdot\mathbf{v}_0 = \rho c^2\mathbf{v}_0 \tag{6}$$

which is merely an eigenproblem for $\mathbf{A}$ with eigenvalue ρc^2 and eigenvector $\mathbf{v}_0$. Because of the assumption of major symmetry on $\mathbf{T}$, the acoustic tensor is symmetric with real eigenvalues. Initially $\mathbf{A}$ is positive definite. During a loading process, the evolution of state variables results in modifications to the tangent tensor and the acoustic tensor, with the result that the lowest eigenvalue of $\mathbf{A}$ decreases. When this eigenvalue obtains a value of zero, material stability is lost and the differential equation for the perturbed motion (3) changes type. Alternatively, one can say that ellipticity of the spatial differential operator is lost.

The eigenvector obtained from eqn (6) associated with the zero eigenvalue is called a discontinuous bifurcation mode, $\mathbf{m}$, because it can be shown that the jump in the velocity gradient and the strain rate are proportional to $\mathbf{m}\otimes\mathbf{n}$ and $\frac{1}{2}(\mathbf{m}\otimes\mathbf{n}+\mathbf{n}\otimes\mathbf{m})$, respectively.

In the past, the analytical determination of the critical case associated with loss of ellipticity has provided a rather complicated set of equations whose solutions were not readily apparent. The next subsection provides a derivation in which it is shown that the equations can be cast in a form readily amenable to a general solution.

2.2. *Specific criteria*

Initially, $\mathbf{T}$ is the elastic tensor, $\mathbf{E}$, and the corresponding acoustic tensor is

$$\mathbf{A}^e = \mathbf{n}\cdot\mathbf{E}\cdot\mathbf{n} \tag{7}$$

which is positive definite. As inelastic deformation occurs, the lowest eigenvalue of $\mathbf{A}$ decreases from the value obtained from $\mathbf{A}^e$. A large number of evolution equations for plasticity and continuum damage provide a tangent tensor consisting of a rank one modification to the elasticity tensor which is of the general form

$$\mathbf{T} = \mathbf{E} - C\mathbf{Q}\otimes\mathbf{Q} \quad \mathbf{Q}:\mathbf{Q} = 1 \tag{8}$$

in which the material parameter, C, and the unit tensor, $\mathbf{Q}$, depend on the deformation path and reflect the current state variables of the particular theory under consideration. The acoustic tensor associated with the tangent tensor is

$$\mathbf{A} = \mathbf{A}^e - C\mathbf{q}\otimes\mathbf{q} \quad \mathbf{q} = \mathbf{Q}\cdot\mathbf{n}. \tag{9}$$

The bifurcation analysis consists of determining the load parameter and the direction $\mathbf{n}$ which first yields zero for the lowest eigenvalue of $\mathbf{A}$. As shown next, if the elasticity tensor is isotropic then analytical expressions can be obtained for both $\mathbf{n}$ and the eigenvalues of $\mathbf{A}$, so that specific conclusions can be made concerning the initiation of failure.

Let $\mathbf{I}$ and $\mathbf{i}$ denote the symmetric fourth-order identity and the second-order identity tensors, respectively. The spherical and deviatoric projections are defined as follows:

$$\mathbf{P}^{sp} = \tfrac{1}{3}\mathbf{i} \otimes \mathbf{i} \quad \mathbf{P}^{dev} = \mathbf{I} - \mathbf{P}^{sp}. \tag{10}$$

Then the isotropic elasticity tensor becomes

$$\mathbf{E} = 3B\mathbf{P}^{sp} + 2G\mathbf{P}^{dev} \tag{11}$$

in which B and G denote the bulk and shear moduli. The acoustic tensors for the elastic and general states are

$$\mathbf{A}^e = \left[B + \frac{G}{3}\right]\mathbf{n} \otimes \mathbf{n} + G\mathbf{i}$$

$$\mathbf{A} = \mathbf{A}^e - C\mathbf{q} \otimes \mathbf{q}. \tag{12}$$

If $\mathbf{q}$ is parallel to $\mathbf{n}$, it follows that $\mathbf{n}$ is an eigenvector of $\mathbf{A}$ with eigenvalue $B + (4G/3) - C(\mathbf{q} \cdot \mathbf{n})^2$. The other eigenvectors are perpendicular to $\mathbf{n}$ with identical eigenvalues of G.

Now suppose $\mathbf{q}$ is not parallel to $\mathbf{n}$. One eigenvector of $\mathbf{A}$ is a vector perpendicular to both $\mathbf{n}$ and $\mathbf{q}$, with eigenvalue $\Lambda_3 = G$. Consider another eigenvector to be the linear combination

$$\mathbf{e} = \alpha\mathbf{n} + \beta\mathbf{q}. \tag{13}$$

The substitution of this equation in the eigenproblem, $\mathbf{A} \cdot \mathbf{e} = \Lambda\mathbf{e}$, and equating coefficients of $\mathbf{n}$ and $\mathbf{q}$ yields

$$\left(\Lambda - B - \frac{4G}{3}\right)\alpha - \left(B + \frac{G}{3}\right)(\mathbf{q} \cdot \mathbf{n})\beta = 0$$

$$C(\mathbf{q} \cdot \mathbf{n})\alpha + \{\Lambda - G + C(\mathbf{q} \cdot \mathbf{q})\}\beta = 0. \tag{14}$$

The existence of a nontrivial solution for α and β yields a quadratic characteristic equation in Λ with solutions Λ_1 and Λ_2. The determinant of the acoustic tensor is $D_A\Lambda_3$ in which

$$D_A = \Lambda_1\Lambda_2 = G\left(B + \frac{4G}{3}\right) - C\Psi$$

$$\Psi = \left(B + \frac{4G}{3}\right)(\mathbf{n} \cdot \mathbf{Q} \cdot \mathbf{Q} \cdot \mathbf{n}) - \left(B + \frac{G}{3}\right)(\mathbf{n} \cdot \mathbf{Q} \cdot \mathbf{n})^2. \tag{15}$$

Ellipticity is lost when D_A first equals zero. The procedure is to minimize D_A (or maximize Ψ) with respect to $\mathbf{n}$, and then obtain the critical value of C by setting D_A to zero:

$$C_{cr} = G\left(B + \frac{4G}{3}\right)\Big/\Psi_{max}. \tag{16}$$

2.3. *Solutions for general case*

Let $\mathbf{e}_i$, $i = 1$–3, be an orthonormal basis so that $\mathbf{n} = n_i\mathbf{e}_i$, summation implied on repeated indices. To obtain the maximum of Ψ let $n_3^2 = 1 - n_1^2 - n_2^2$ and set the derivatives of Ψ with respect to n_1 and n_2 equal to zero. The result is the nonlinear set of equations

$$\frac{\partial \Psi}{\partial \mathbf{n}} \cdot \frac{\partial \mathbf{n}}{\partial n_1} = 0 \quad \frac{\partial \Psi}{\partial \mathbf{n}} \cdot \frac{\partial \mathbf{n}}{\partial n_2} = 0. \tag{17}$$

The individual terms in these equations are

$$\frac{1}{2}\frac{\partial \Psi}{\partial \mathbf{n}} = \left(B+\frac{4G}{3}\right)(\mathbf{n}\cdot\mathbf{Q}\cdot\mathbf{Q}) - 2\left(B+\frac{G}{3}\right)(\mathbf{n}\cdot\mathbf{Q})$$

$$\frac{\partial \mathbf{n}}{\partial n_1} = \mathbf{e}_1 - \frac{n_1}{n_3}\mathbf{e}_3 \quad \frac{\partial \mathbf{n}}{\partial n_2} = \mathbf{e}_2 - \frac{n_2}{n_3}\mathbf{e}_3. \tag{18}$$

With $\mathbf{Q} = Q_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, it follows that

$$\mathbf{n}\cdot\mathbf{Q}\cdot\frac{\partial \mathbf{n}}{\partial n_1} = n_i\left(Q_{i1} - \frac{n_1}{n_3}Q_{i3}\right)$$

$$\mathbf{n}\cdot\mathbf{Q}\cdot\mathbf{Q}\cdot\frac{\partial \mathbf{n}}{\partial n_1} = n_i Q_{ij}\left(Q_{j1} - \frac{n_1}{n_3}Q_{j3}\right). \tag{19}$$

When eqn (19) and the corresponding expressions involving the derivative with respect to n_2 are substituted into eqn (17), the result is a very complicated nonlinear set of equations for determining n_1 and n_2. However, suppose that $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are chosen to be the principal basis of $\mathbf{Q}$ with eigenvalues Q_1, Q_2 and Q_3. Then eqn (19) reduces to

$$\mathbf{n}\cdot\mathbf{Q}\cdot\frac{\partial \mathbf{n}}{\partial n_1} = n_1(Q_1 - Q_3)$$

$$\mathbf{n}\cdot\mathbf{Q}\cdot\mathbf{Q}\frac{\partial \mathbf{n}}{\partial n_1} = n_1(Q_1^2 - Q_3^2) = n_1(Q_1 - Q_3)(Q_1 + Q_3). \tag{20}$$

The result is that eqn (17) reduces to the nonlinear set of equations

$$n_1(Q_1 - Q_3)\left[\left(B+\frac{4G}{3}\right)(Q_1+Q_3) - 2\left(B+\frac{G}{3}\right)(\mathbf{n}\cdot\mathbf{Q}\cdot\mathbf{n})\right] = 0 \tag{21a}$$

$$n_2(Q_2 - Q_3)\left[\left(B+\frac{4G}{3}\right)(Q_2+Q_3) - 2\left(B+\frac{G}{3}\right)(\mathbf{n}\cdot\mathbf{Q}\cdot\mathbf{n})\right] = 0 \tag{21b}$$

which is much simpler than the form obtained without the use of the principal basis.

There are two groups of solutions to eqns (21a) and (21b). Assume the eigenvalues Q_1, Q_2, Q_3 are distinct. One solution for Group 1 is obtained by merely choosing the appropriate components of $\mathbf{n}$ to be zero; i.e. $n_1 = 0$ and $n_2 = 0$ with the implication that $n_3 = 1$. Then $\mathbf{n}\cdot\mathbf{Q}\cdot\mathbf{Q}\cdot\mathbf{n} = Q_3^2$, $\mathbf{n}\cdot\mathbf{Q}\cdot\mathbf{n} = Q_3$ and $\Psi_{\max} = GQ_3^2$. Other solutions are obtained by permuting indices.

One solution for Group 2 consists of setting one of the components of $\mathbf{n}$ to zero to force one of the equations (21a) or (21b) to be zero, and the square bracket is set to zero in the remaining equation. Specifically, suppose $n_1 = 0$ and the quantity in square brackets in eqn (21b) is zero or

$$\mathbf{n}\cdot\mathbf{Q}\cdot\mathbf{n} = \frac{\left(B+\frac{4G}{3}\right)(Q_2+Q_3)}{2\left(B+\frac{G}{3}\right)} \tag{22}$$

which can be substituted in the expression for Ψ in eqn (15) to obtain Ψ_{max} and consequently the critical value of C. Also, for this case $\mathbf{n}\cdot\mathbf{Q}\cdot\mathbf{n} = (Q_2-Q_3)n_2^2+Q_3$ and eqn (22) is used to solve for n_2. Again, two other solutions are obtained by permuting indices.

For convenience, convert to the following dimensionless variables:

$$\hat{B} = \frac{3B}{2G} = \frac{1+v}{1-2v} \quad \hat{C} = \frac{C}{2G} \tag{23}$$

in which v denotes Poisson's ratio. $\hat{B}$ varies from a value of 1 when $v = 0$ to infinity when $v = 0.5$. These are the two dimensionless material parameters which appear in the criteria for loss of material stability.

The bifurcation solutions are summarized as follows:

Group 1:

$$\text{(i)} \quad n_1 = 1, n_2 = 0, n_3 = 0 \quad \hat{C}_1 = \frac{(\hat{B}+2)}{3Q_1^2} \tag{24a}$$

$$\text{(ii)} \quad n_1 = 0, n_2 = 1, n_3 = 0 \quad \hat{C}_2 = \frac{(\hat{B}+2)}{3Q_2^2} \tag{24b}$$

$$\text{(iii)} \quad n_1 = 0, n_2 = 0, n_3 = 1 \quad \hat{C}_3 = \frac{(\hat{B}+2)}{3Q_3^2}. \tag{24c}$$

Group 2:

(i) $n_1 = 0$

$$n_2^2 = \frac{(\hat{B}+2)Q_2-(\hat{B}-1)Q_3}{(2\hat{B}+1)(Q_2-Q_3)} \quad n_3^2 = 1-n_2^2$$

$$\hat{C}_4 = \frac{(2\hat{B}+1)}{(\hat{B}+2)(Q_2^2+Q_3^2)-2(\hat{B}-1)Q_2Q_3}. \tag{25a}$$

(ii) $n_2 = 0$

$$n_3^2 = \frac{(\hat{B}+2)Q_3-(\hat{B}-1)Q_1}{(2\hat{B}+1)(Q_3-Q_1)} \quad n_1^2 = 1-n_3^2$$

$$\hat{C}_5 = \frac{(2\hat{B}+1)}{(\hat{B}+2)(Q_3^2+Q_1^2)-2(\hat{B}-1)Q_3Q_1}. \tag{25b}$$

(iii) $n_3 = 0$

$$n_1^2 = \frac{(\hat{B}+2)Q_1-(\hat{B}-1)Q_2}{(2\hat{B}+1)(Q_1-Q_2)} \quad n_2^2 = 1-n_1^2$$

$$\hat{C}_6 = \frac{(2\hat{B}+1)}{(\hat{B}+2)(Q_1^2+Q_2^2)-2(\hat{B}-1)Q_1Q_2}. \tag{25c}$$

The governing or critical case is the one that yields the minimum value: $\hat{C}_{cr} = \min_i \hat{C}_i$. Two solutions for $\mathbf{n}$ that are equal and opposite in sign are always possible and actually define the normal to only one surface. However, if the critical case is in Group 2, then two distinct solutions for $\mathbf{n}$ are possible and such a situation is sometimes observed with crossing shear bands. In other cases, one solution might be ruled out based on a physical consideration of the mode which is discussed later, e.g. shear and compression might not develop, whereas shear with tension would.

2.4. *Solutions for the case of two identical eigenvalues*

For special stress states, e.g. uniaxial stress, triaxial compression, triaxial extension and equal biaxial stress, it is possible that two of the eigenvalues of **Q** will be identical. Suppose $Q_1 = Q_3$. Then, eqn (21a) is automatically satisfied. In Group 1, the first and third cases provide the same values for $\hat{C}_1$ and $\hat{C}_3$. In Group 2, either $n_2 = 0$ or the quantity in the square brackets is zero in eqn (21b). If $n_2 = 0$, then $\mathbf{n}\cdot\mathbf{Q}\cdot\mathbf{Q}\cdot\mathbf{n} = Q_1^2$, $\mathbf{n}\cdot\mathbf{Q}\cdot\mathbf{n} = Q_1$ and the result for $\hat{C}_{cr}$ is the same as one of the cases in Group 1, but with the condition now that $n_1^2+n_3^2 = 1$, i.e. a unique bifurcation plane does not exist. This result is incorporated in Group 1. The remaining solution, obtained by setting the square bracket in eqn (21b) to zero, yields the only solution in which all components of **n** may be nonzero. The results are summarized below for $Q_1 = Q_3$:

Group 1:

$$\text{(i)}\quad n_1 = 1, n_2 = 0, n_3 = 0 \quad \hat{C}_1 = \frac{(\hat{B}+2)}{3Q_1^2} \tag{26a}$$

$$\text{(ii)}\quad n_2 = 0, n_1^2+n_3^2 = 1 \quad \hat{C}_2 = \frac{(\hat{B}+2)}{3Q_2^2}. \tag{26b}$$

Group 2:

$$n_2^2 = \frac{(\hat{B}+2)Q_2-(\hat{B}-1)Q_1}{(2\hat{B}+1)(Q_2-Q_1)} \quad n_1^2+n_3^2 = 1-n_2^2$$

$$\hat{C}_3 = \frac{(2\hat{B}+1)}{(\hat{B}+2)(Q_2^2+Q_1^2)-2(\hat{B}-1)Q_2Q_1}. \tag{26c}$$

If a different pair of eigenvalues are equal, the appropriate expressions for $\hat{C}$ and the components of **n** can be determined similarly.

2.5. *Solutions for the case of three identical eigenvalues*

If all the eigenvalues of **Q** are identical, $Q = Q_1 = Q_2 = Q_3$, then **n** can be any vector and $\hat{C}_{cr} = (\hat{B}+2)/(3Q^2)$.

2.6. *Modes*

The mode, **m**, is defined to be the eigenvector of the acoustic tensor at the critical value of **n** and with an eigenvalue of zero. Therefore **m** is given by

$$\mathbf{A}\cdot\mathbf{m} = 0 \tag{27}$$

or, with the use of eqns (12) and (23),

$$(2\hat{B}+1)(\mathbf{m}\cdot\mathbf{n})\mathbf{n}+3\mathbf{m}-6\hat{C}(\mathbf{q}\cdot\mathbf{m})\mathbf{q} = 0. \tag{28}$$

If the components of **m** are expressed in terms of the principal basis, $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, and eqn (9) is used, then the first component equation of eqn (28) is

$$[(2\hat{B}+1-6\hat{C}Q_1^2)n_1^2+3]m_1+[(2\hat{B}+1-6\hat{C}Q_1Q_2)n_1n_2]m_2 \\ +[(2\hat{B}+1-6\hat{C}Q_1Q_3)n_1n_3]m_3 = 0 \tag{29}$$

with the other equations obtained by a permutation of the indices. A maximum of two of these equations is independent, so the restriction of $\mathbf{m}\cdot\mathbf{m} = 1$ is imposed.

3. GENERAL SOLUTIONS

The expressions for the critical values of the hardening modulus and orientation of the failure band are completely general for the given form of the tangent tensor. The relatively simple forms involve only a few elastic and plastic material variables which suggest that critical conditions can be conveniently displayed in the form of contour plots. Such plots are given in this section.

Since $\mathbf{Q}$ is normalized to be a unit tensor, it follows that the principal values are constrained to be on the unit sphere

$$Q_1^2+Q_2^2+Q_3^2 = 1 \tag{30}$$

so that only two independent variables are required to define $\mathbf{Q}$ once the principal axes are determined. A natural choice for locating points on a sphere are the polar angles ϕ and θ with

$$Q_1 = \sin\phi\cos\theta \quad Q_2 = \sin\phi\sin\theta \quad Q_3 = \cos\phi \quad 0 < \theta \leqslant 2\pi \quad 0 < \phi \leqslant \pi. \tag{31}$$

For easy interpretation of later figures, contour plots of Q_1, Q_2 and Q_3 are given as functions of the polar angles in Fig. 1.

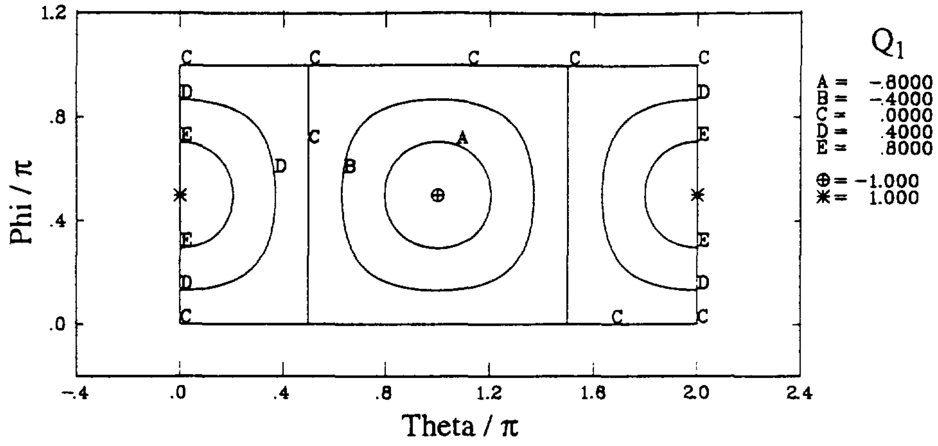
Figure 2a–d shows contour plots of critical values of the dimensionless inelastic parameter, $\hat{C}$, as functions of θ and ϕ. The contours are smooth, but the governing criteria actually come from taking the minimum of the six conditions $\hat{C}_i = \hat{C}_{cr}$, $i = 1,6$. An immediate observation is that the minimum value for $\hat{C}$ is one. For plasticity, this value corresponds to the limit point. Values for $\hat{C}$ greater than one indicate softening is needed to initiate a discontinuous bifurcation. Note that when Poisson's ratio is zero (Fig. 2a) $\hat{C} = 1$ at the poles defined by $\phi = 0, \pi/2, \pi$ and $\theta = 0, \pi/2, \pi, 3\pi/2, 2\pi$. When Poisson's ratio is not equal to zero, $\hat{C}$ obtains a value of one at a limited number of points. A comparison of Figs 1 and 2 shows that $\mathbf{Q}$ has an eigenvalue exactly equal to zero at every point where $\hat{C} = 1$. This implies that a very special stress state must exist if a discontinuous bifurcation is to occur at the limit point. Materials with small Poisson's ratio always exhibit a discontinuous bifurcation for small amounts of softening ($\hat{C}$ close to one), whereas considerable softening may be required for large values of Poisson's ratio.

The numerical results indicate that the maximum value for the inelastic coefficient, $\hat{C}$, occurs at just two points; $Q_1 = Q_2 = Q_3 = 1/\sqrt{3}$ and $Q_1 = Q_2 = Q_3 = -(1/\sqrt{3})$ for Poisson's ratio $\neq 0$. If positive values for the eigenvalues are selected, then $\theta = \pi/4$ and $\sin^2\phi = 2/3$ or $\phi = 54.7°$. When Poisson's ratio is zero, then the maximum is achieved at eight points: $Q_1^2 = Q_2^2 = Q_3^2 = \frac{1}{3}$. For any one of these cases, all three of the criteria in Group 2 are satisfied and the inelastic coefficient achieves its maximum possible value of $\hat{C}_{max} = 0.5+\hat{B} = 3/2(1-2v)$ for any value of Poisson's ratio. If a particular model yields values of $\hat{C}$ up to this maximum, then a discontinuous bifurcation will occur no matter what state path is followed.

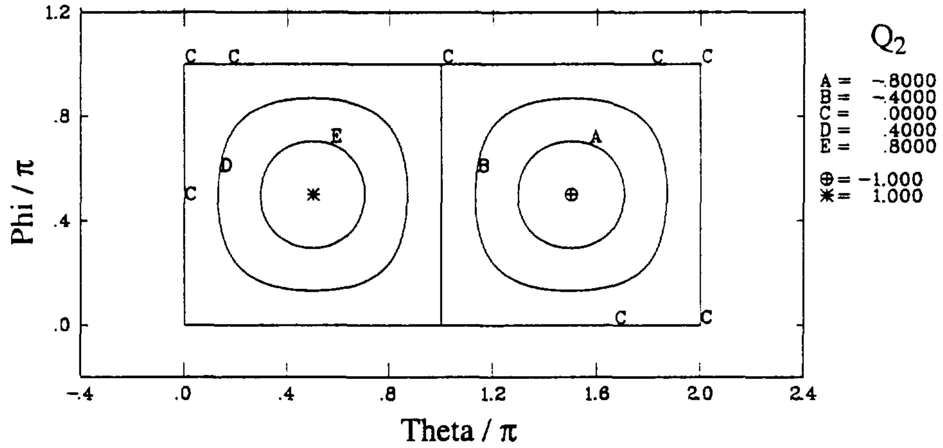
Many materials are governed by evolution equations with the restriction that $\mathbf{Q}$ be a deviatoric tensor, or $Q_1+Q_2+Q_3 = 0$. The same restriction defined in terms of polar angles is the following:

$$\cos\phi+\sin\phi(\sin\theta+\cos\theta) = 0 \tag{32}$$

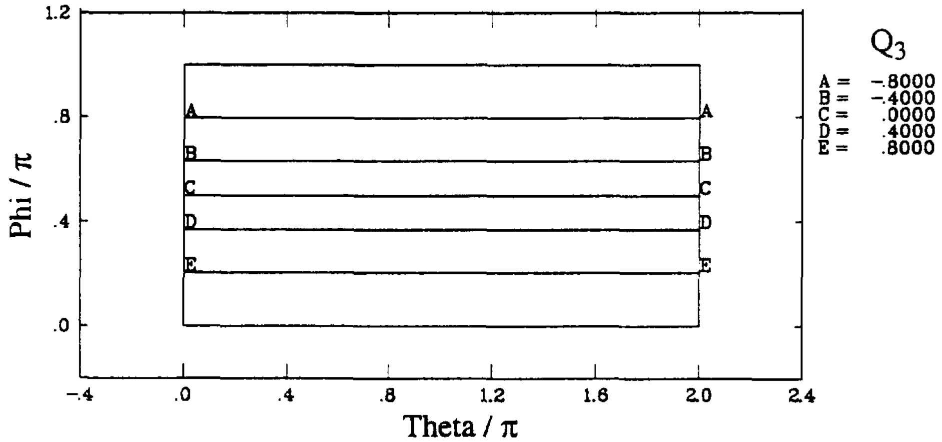
which is shown as a dotted line on each of the plots of Fig. 2. For values of Poisson's ratio not equal to zero, the deviatoric line passes through all the limit points of $\hat{C} = 1$. An immediate observation is that when Poisson's ratio is not zero and $\mathbf{Q}$ is not a deviatoric tensor, then a discontinuous bifurcation cannot occur at the limit point.



(a) Q_1

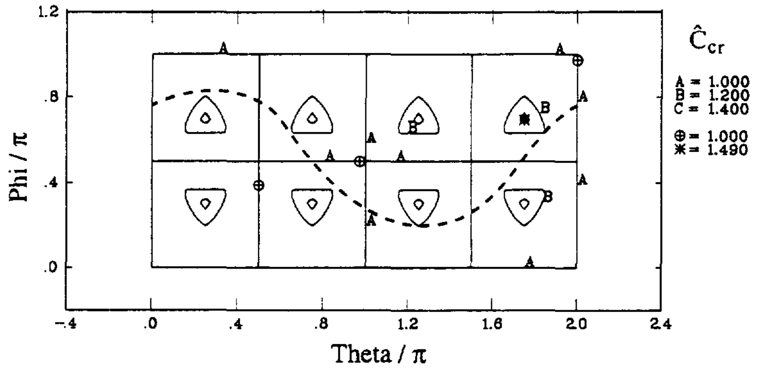


(b) Q_2

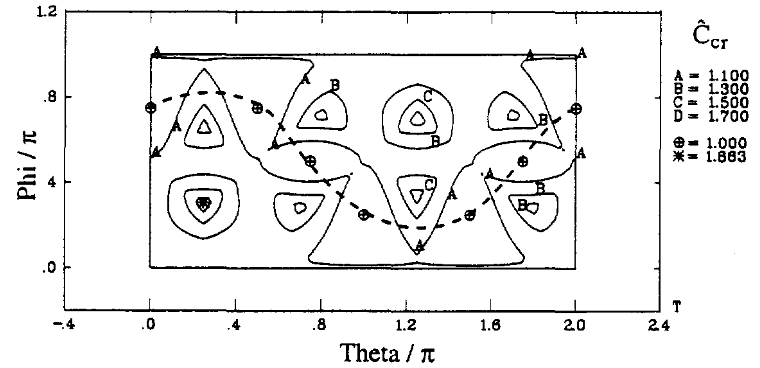


(c) Q_3

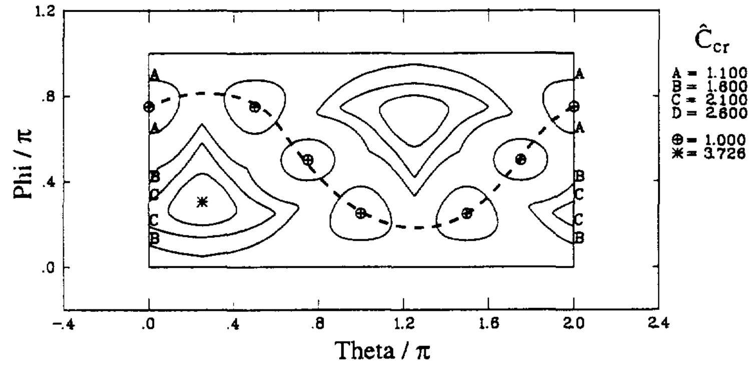
Fig. 1. Contour plots of (a), Q_1, (b) Q_2 and (c) Q_3 as functions of polar angles.



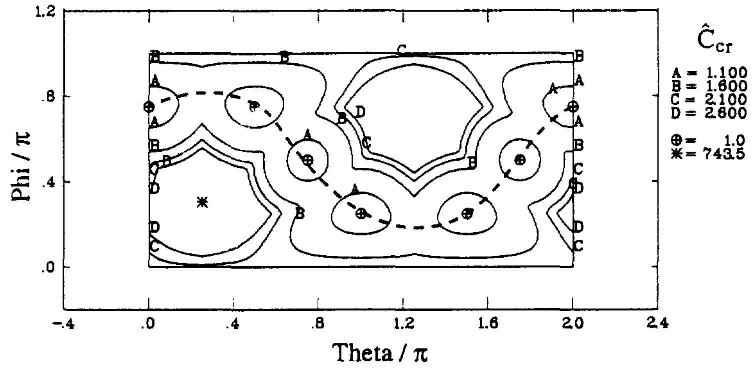
(a) Poisson's Ratio = 0.0



(b) Poisson's Ratio = 0.1



(c) Poisson's Ratio = 0.3



(d) Poisson's Ratio = 0.499

Fig. 2. Contour lines for critical values of $\hat{C}$ as functions of polar angles with the deviatoric restriction shown as a dotted line. (a) $\nu = 0$ or $\hat{B} = 1$; (b) $\nu = 0.1$ or $\hat{B} = 1.375$; (c) $\nu = 0.3$ or $\hat{B} = 3.25$; (d) $\nu = 0.499$ or $\hat{B} = 749.5$.

Figure 3a–d shows the domain of application for each criterion. Only for $v = 0$ do the first three criteria appear and, for each case, the domain is small. The result is that for all practical purposes only the three criteria of Group 2 govern and for numerical solutions it would not be necessary to check the three criteria of Group 1. These plots can also be used in conjunction with eqns (25a), (25b) and (25c) to deduce the direction $\mathbf{n}$, since criteria 4–6 are associated with $n_1 = 0$, $n_2 = 0$ and $n_3 = 0$, respectively. The actual values for the other two components must be evaluated for each case. Note that, in general, two signs for the square root implies two solutions for $\mathbf{n}$ for each criterion.

In addition to the deviatoric condition, other restrictions such as $Q_1 = Q_2$ can be easily superposed on Figs 2 and 3 just by determining the equation for the restriction in polar coordinates. In summary, these plots provide a convenient method for interpreting the features of a discontinuous bifurcation once the eigensystem of $\mathbf{Q}$ has been determined.

4. SPECIFIC CONSTITUTIVE EQUATIONS

Here, formulations for constitutive equations based on conventional plasticity and continuum damage are given to show that the general form of eqn (8) is obtained and that the results of the previous section can be used for such models.

4.1. *Plasticity*

Let f denote a yield function which depends on the stress, $\boldsymbol{\sigma}$, the back stress, $\boldsymbol{\sigma}^B$, and the hardening stress, $\bar{\sigma}$. Suppose the yield function has been constructed in the usual manner so that $f < 0$ denotes the elastic state, plasticity occurs if $f = 0$, and $f > 0$ is not allowed. Let the evolution of plasticity be described with the use of a monotonically increasing parameter, λ, and associated evolution equations for plastic strain, $\mathbf{e}^p$, and effective plastic strain, $\bar{e}^p$:

$$\dot{\mathbf{e}}^p = \dot{\lambda}\mathbf{N} \quad \dot{\bar{e}}^p = \dot{\lambda}\bar{n}$$
$$\mathbf{N} = \frac{\partial f}{\partial \boldsymbol{\sigma}} \quad \bar{n} = -\frac{\partial f}{\partial \bar{\sigma}} \tag{33}$$

with the minus sign introduced to provide the usual correspondence between $\dot{\bar{e}}^p$ and $\dot{\lambda}$ for hardening materials. If the back stress enters the yield function only in the form $f[(\boldsymbol{\sigma} - \boldsymbol{\sigma}^B), \bar{\sigma}]$, then

$$\frac{\partial f}{\partial \boldsymbol{\sigma}^B} = -\frac{\partial f}{\partial \boldsymbol{\sigma}} = -\mathbf{N} \tag{34}$$

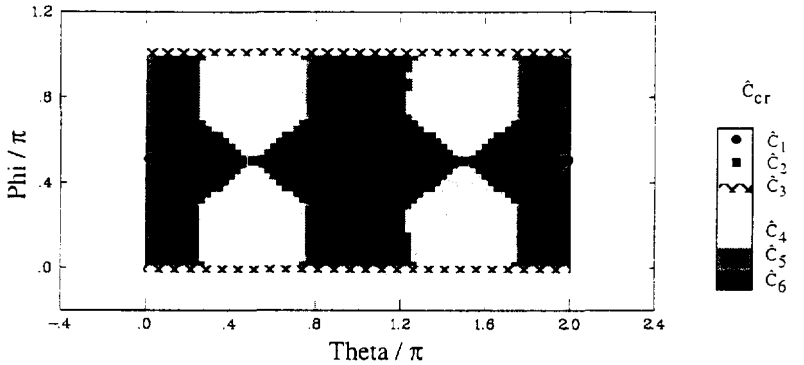
and the consistency condition for continued plastic deformation, $\dot{f} = 0$, yields

$$\mathbf{N}:(\dot{\boldsymbol{\sigma}} - \dot{\boldsymbol{\sigma}}^B) - \bar{n}\dot{\bar{\sigma}} = 0. \tag{35}$$

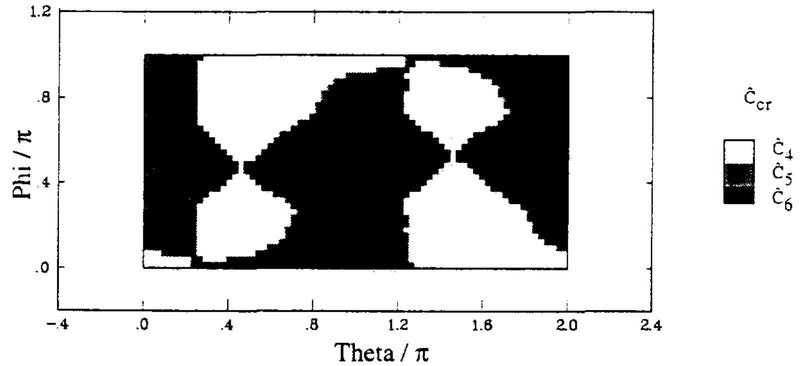
Introduce constitutive equations for small deformations of the following form:

$$\dot{\boldsymbol{\sigma}} - \dot{\boldsymbol{\sigma}}^B = \mathbf{E}:(\dot{\mathbf{e}} - \dot{\mathbf{e}}^p) \quad \dot{\boldsymbol{\sigma}}^B = \mathbf{B}:\dot{\mathbf{e}}^p \quad \dot{\bar{\sigma}} = h\dot{\bar{e}}^p \tag{36}$$

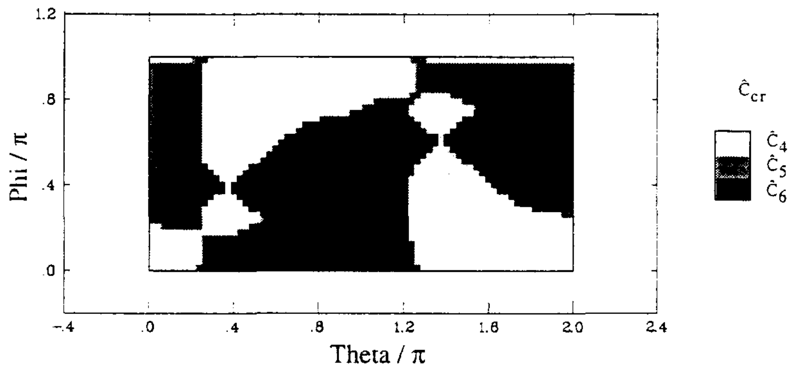
in which $\mathbf{B}$ denotes a fourth-order tensor and h, a material parameter. Then the consistency condition (34) yields a specific equation for the rate of increase of the plasticity parameter:



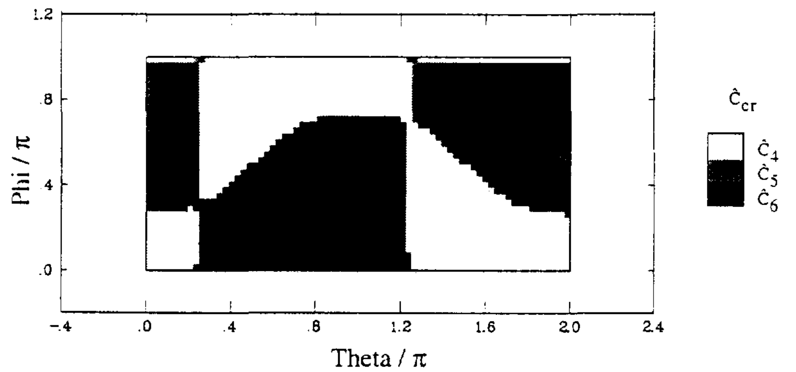
(a) Poisson's Ratio = 0.0



(b) Poisson's Ratio = 0.1



(c) Poisson's Ratio = 0.3



(d) Poisson's Ratio = 0.499

Fig. 3. Domain for each criterion as a function of polar angles. (a) $\nu = 0$ or $\hat{B} = 1$; (b) $\nu = 0.1$ or $\hat{B} = 1.375$; (c) $\nu = 0.3$ or $\hat{B} = 3.25$; (d) $\nu = 0.499$ or $\hat{B} = 749.5$.

$$\dot{\lambda} = \frac{\mathbf{N}:\mathbf{E}:\dot{\mathbf{e}}}{K+H} \quad K = \mathbf{N}:\mathbf{E}:\mathbf{N} \quad H = h\bar{n}^2 \tag{37}$$

with H called the hardening modulus for the plasticity model. To obtain the tangent tensor, use the relations

$$\dot{\boldsymbol{\sigma}} = \mathbf{E}:(\dot{\mathbf{e}}-\dot{\mathbf{e}}^{\mathrm{p}})+\mathbf{B}:\dot{\mathbf{e}}^{\mathrm{p}} = \mathbf{T}:\dot{\mathbf{e}} \tag{38}$$

to obtain

$$\mathbf{T} = \mathbf{E} - \frac{1}{K+H}(\mathbf{E}-\mathbf{B}):\mathbf{N}\otimes\mathbf{N}:\mathbf{E} \tag{39}$$

which is not symmetric in general. However, if the evolution equation for the back stress is modified so that $\mathbf{B} = \beta\mathbf{E}$ with $0 \leqslant \beta < 1$, then

$$\mathbf{T} = \mathbf{E} - C\mathbf{Q}\otimes\mathbf{Q} \tag{40}$$

in which

$$\mathbf{Q} = \frac{\mathbf{Q}^*}{Q^*} \quad \mathbf{Q}^* = \mathbf{E}:\mathbf{N} \quad Q^* = (\mathbf{Q}^*:\mathbf{Q}^*)^{1/2} \quad C = \frac{(1-\beta)Q^{*2}}{K+H}. \tag{41}$$

As a specific example, consider von Mises plasticity with isotropic hardening ($\beta = 0$) and a yield function given by

$$f = (\tfrac{1}{2}\boldsymbol{\sigma}^{\mathrm{d}}:\boldsymbol{\sigma}^{\mathrm{d}})^{1/2} - \bar{\sigma} \quad \bar{\sigma} = \bar{\sigma}_0 + h\bar{e}^{\mathrm{p}} \tag{42}$$

in which $\bar{\sigma}_0$ denotes the initial yield stress, and $\boldsymbol{\sigma}^{\mathrm{d}}$ the stress deviator. Then

$$\mathbf{N} = \frac{\frac{1}{2}\boldsymbol{\sigma}^{\mathrm{d}}}{(\frac{1}{2}\boldsymbol{\sigma}^{\mathrm{d}}:\boldsymbol{\sigma}^{\mathrm{d}})^{1/2}} \quad \bar{n} = 1 \tag{43}$$

and

$$K = G \quad H = h. \tag{44}$$

It follows that

$$\mathbf{Q}^* = \frac{G\boldsymbol{\sigma}^{\mathrm{d}}}{(\frac{1}{2}\boldsymbol{\sigma}^{\mathrm{d}}:\boldsymbol{\sigma}^{\mathrm{d}})^{1/2}} \quad \mathbf{Q} = \frac{\boldsymbol{\sigma}^{\mathrm{d}}}{(\boldsymbol{\sigma}^{\mathrm{d}}:\boldsymbol{\sigma}^{\mathrm{d}})^{1/2}} \quad Q^{*2} = 2G^2 \tag{45}$$

and

$$C = \frac{2G^2}{G+h} \quad \hat{C} = \frac{G}{G+h}. \tag{46}$$

The critical hardening modulus becomes

$$h_{\mathrm{cr}} = \frac{G}{\hat{C}_{\mathrm{cr}}}(1-\hat{C}_{\mathrm{cr}}). \tag{47}$$

As shown in the previous section, the lowest possible value for $\hat{C}_{\mathrm{cr}}$ is one in which case the critical hardening, h_{cr}, modulus is zero. Otherwise, the critical hardening modulus is negative, a result that is well-known for associated isotropic plasticity.

The von Mises model is identified with the case where **N** and, consequently, **Q** are deviatoric tensors. As seen in Fig. 2, the deviatoric condition is satisfied along a single line. Points along this line are identified uniquely by specifying θ. Figure 4 provides plots of Q_1, Q_2, Q_3, and critical values of $\hat{C}$ as a function of θ for various values of Poisson's ratio. One immediate observation from the plots is that, for $\nu > 0$, the limit state condition of $\hat{C} = 1$ occurs only when one of the eigenvalues of **Q** is zero, a state of stress which can be considered one of pure shear. This result was noted previously by Neilsen and Schreyer (1993). The value of $\hat{C}$, the amount of softening needed to initiate a discontinuous bifurcation, increases with an increase in Poisson's ratio. $\hat{C}$ attains a maximum value when the eigenvalues of **Q** are $1/\sqrt{6}(2; -1; -1)$, which, for example, are generated by uniaxial stress. These results indicate that a von Mises plasticity model will exhibit a discontinuous bifurcation at the limit point when the material is subjected to pure shear, but will not exhibit a discontinuous bifurcation in uniaxial tension until the material is loaded well into the softening regime.

4.2. *Elementary damage models*

Here, the tangent tensor is developed for a continuum damage model with a thermodynamically consistent approach suggested originally by Simo and Hughes (in press). Suppose the Helmholtz free energy under isothermal conditions is given by a stored elastic energy term and a term for the energy of free surfaces created by cracks:

$$H = \tfrac{1}{2}\mathbf{e}:\mathbf{E}:\mathbf{e} + \tfrac{1}{2}A(\bar{c}) \tag{48}$$

in which $\bar{c}$ denotes the average crack radius and A is a function left arbitrary here. As before, **e** is the strain but the elasticity tensor is allowed to change as the material damages. The first law of thermodynamics implies that the stress continues to be given by $\mathbf{s} = \mathbf{E}:\mathbf{e}$ and, in addition, provides the motivation for defining the rate of dissipation as follows:

$$D_s = -\tfrac{1}{2}\mathbf{e}:\dot{\mathbf{E}}:\mathbf{e} - \tfrac{1}{2}h\dot{\bar{c}} \quad h = \frac{\partial A}{\partial \bar{c}}. \tag{49}$$

If the principal of maximum dissipation is invoked, then evolution equations for the elasticity tensor and the average crack radius are proportional to their respective conjugate forces:

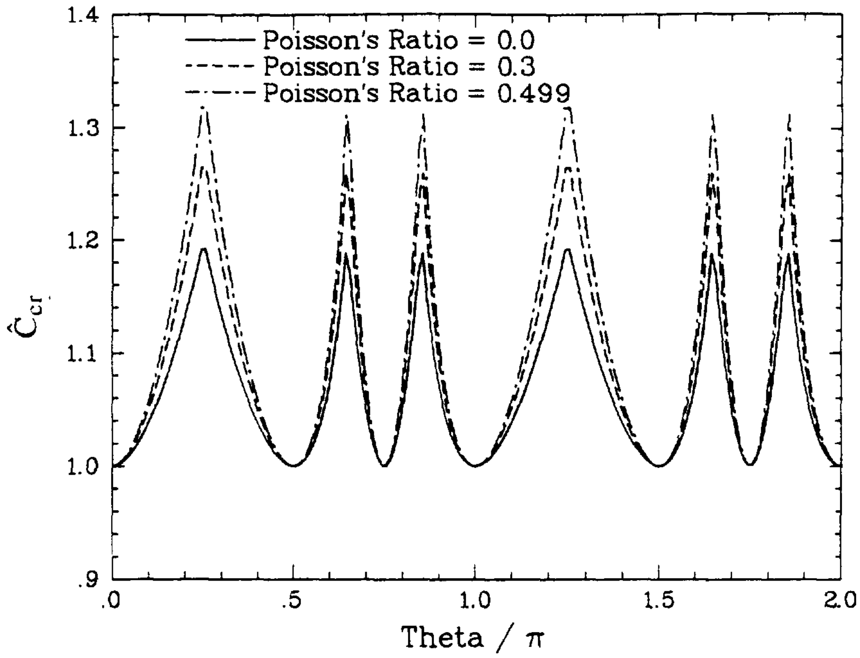
$$\dot{\mathbf{E}} = -\dot{\omega} c_E \mathbf{e} \otimes \mathbf{e} \quad \dot{\bar{c}} = \dot{\omega} h \tag{50}$$

in which ω is a monotonically increasing parameter used to parameterize the equations governing the damage process and c_E is a material parameter introduced for dimensional consistency. The use of the first of eqn (50) results in a tangent tensor that is a rank one modification to the current elasticity tensor which will not be isotropic. To utilize the formulation of the previous sections, it is necessary to maintain isotropy of the elasticity tensor, a property not exhibited by the continuum damage formulation given above.

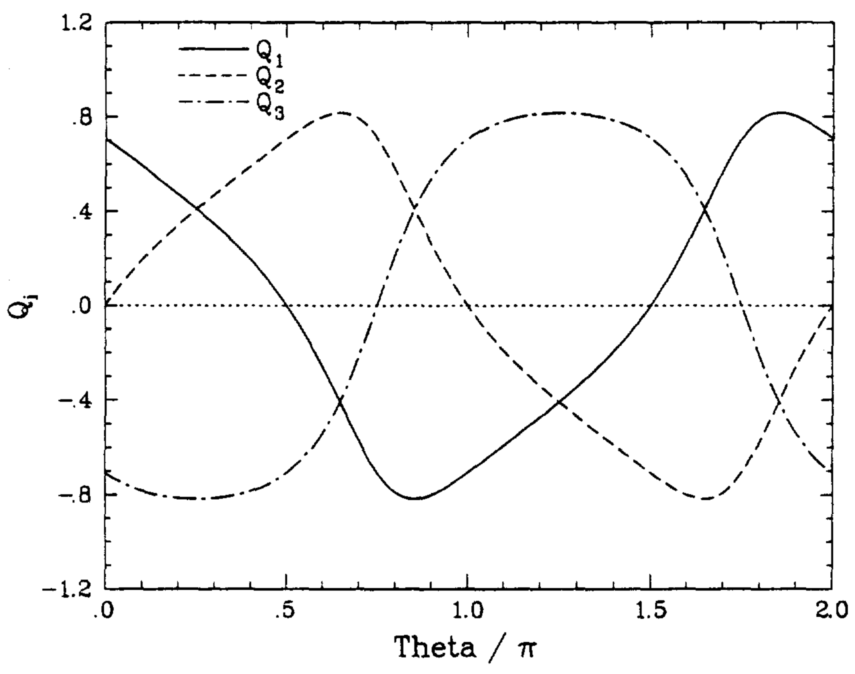
If isotropic damage is assumed, then eqn (11) must hold with B and G considered as variables. The elastic stored energy becomes

$$\tfrac{1}{2}\mathbf{e}:\mathbf{E}:\mathbf{e} = \tfrac{1}{2}[3B(\bar{e}_s)^2 + 2G(\bar{e}_d)^2] \tag{51}$$

in which



(a)



(b)

Fig. 4. Plots of Q_1, Q_2, Q_3 and critical values of $\hat{C}$ as a function of the polar angle θ for the von Mises plasticity model.

$$\bar{e}_s^2 = \mathbf{P}^{sp} :: (\mathbf{e} \otimes \mathbf{e}) = \tfrac{1}{3}(e_v)^2 \quad \bar{e}_d^2 = \mathbf{P}^{dev} :: (\mathbf{e} \otimes \mathbf{e}) = \mathbf{e}^d : \mathbf{e}^d. \tag{52}$$

The result of invoking the principal of maximum dissipation now yields

$$\dot{B} = -\dot{\omega} c_B \bar{e}_s^2 \quad \dot{G} = -\dot{\omega} c_G \bar{e}_d^2 \tag{53}$$

with $c_B = 1$ and $c_G = 1$. However, microstructural arguments often provide constraints in the sense that the bulk and shear moduli often do not damage at the same rate, or, one of these parameters may not damage at all. The introduction of the coefficients c_B and c_G allows for such constraints. In actual fact, damage is generally anisotropic so even this formulation is severely limited.

The dissipation expression of eqn (49) becomes

$$D_s = \tfrac{1}{2}\dot{\omega}[3c_B \bar{e}_s^4 + 2c_G \bar{e}_d^4 - h^2] \tag{54}$$

which suggests that a damage function be defined with the use of the terms in square brackets:

$$f_D = 3c_B \bar{e}_s^4 + 2c_G \bar{e}_d^4 - h^2 - f_{Do} \tag{55}$$

where f_{Do} is a positive constant. Suppose damage does not occur if $f_D < 0$, the situation of $f_D > 0$ is not allowed, and damage can occur only if $f_D = 0$. Then eqns (54) and (55) indicate that the dissipation rate is

$$D_s = \tfrac{1}{2}\dot{\omega} f_{Do}. \tag{56}$$

Since $D_s \geqslant 0$, the second law of thermodynamics is met.

The application of the consistency condition, $\dot{f}_D = 0$, to eqn (55) yields

$$3c_B \bar{e}_s^2(\mathbf{e} : \mathbf{P}^{sp} : \dot{\mathbf{e}}) + 2c_G \bar{e}_d^2(\mathbf{e} : \mathbf{P}^{dev} : \dot{\mathbf{e}}) - \frac{1}{2}\frac{\partial h}{\partial \bar{c}}\dot{\bar{c}} = 0 \tag{57}$$

which, with the use of eqn (50), yields

$$\dot{\omega} = \frac{[3c_B \bar{e}_s^2 \mathbf{e} : \mathbf{P}^{sp} + 2c_G \bar{e}_d^2 \mathbf{e} : \mathbf{P}^{dev}] : \dot{\mathbf{e}}}{\tfrac{1}{2} h \dfrac{\partial h}{\partial \bar{c}}}. \tag{58}$$

Consider the equation for the stress rate:

$$\dot{\mathbf{s}} = \mathbf{E} : \dot{\mathbf{e}} + \dot{\mathbf{E}} : \mathbf{e} = \mathbf{T} : \dot{\mathbf{e}}. \tag{59}$$

The use of eqn (53) implies that the evolution equation for the elasticity tensor is

$$\dot{\mathbf{E}} = -\dot{\omega}[3c_B \bar{e}_s^2 \mathbf{P}^{sp} + 2c_G \bar{e}_d^2 \mathbf{P}^{dev}]. \tag{60}$$

The substitution of eqns (58) and (60) into eqn (59) yields

$$\mathbf{T} = \mathbf{E} - C^d \mathbf{Q} \otimes \mathbf{Q} \tag{61}$$

with

$$\mathbf{Q} = \frac{\mathbf{Q}^*}{Q^*} \qquad \mathbf{Q}^* = 3c_{\mathrm{B}}\bar{e}_{\mathrm{s}}^3\mathbf{i} + 2c_{\mathrm{G}}\bar{e}_{\mathrm{d}}^2\mathbf{e}^{\mathrm{d}}$$

$$C^{\mathrm{d}} = \frac{Q^{*2}}{\frac{1}{2}h\frac{\partial h}{\partial \bar{c}}} \quad Q^{*2} = 3(3c_{\mathrm{B}}\bar{e}_{\mathrm{s}}^3)^2 + (2c_{\mathrm{G}})^2\bar{e}_{\mathrm{d}}^6. \tag{62}$$

The important difference now is that, in addition to changes in C^{d}, the bulk and shear moduli are changing so the bifurcation maps for different values of dimensionless bulk moduli (or Poisson's ratio) must be considered.

5. SUMMARY

A possible criterion for the initiation of material failure is the equivalent criterion of the existence of a discontinuous bifurcation, of the appearance of material instability, or of a change in type of the differential equation of motion of a continuum. If the tangent tensor is a rank-one modification to the isotropic elastic tensor, then it is possible to obtain general solutions for the critical parameters at the onset of failure, for the direction of the normal to the failure zone, and for the mode of failure. Because of the limited number of free parameters, the results for critical values of dimensionless material parameters are presented in terms of contour plots. In conjunction with a specific constitutive equation, such plots can provide qualitative information concerning the existence or nonexistence of bifurcation states for specific stress paths. Conversely, if experimental data are available, the plots can be used to illustrate bifurcation features that must be reflected by a proposed constitutive equation.

Although the analysis is limited to isotropic elasticity with the rank one modification for the tangent tensor, the results hold for the majority of plasticity and damage models currently used in engineering practice.

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